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Characters for $\widehat{\mathfrak{sl}(n)}_{k=1}$ from a novel thermodynamic Bethe ansatz*

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Abstract. Motivated by the recent development on the exact thermodynamics of one-dimensional quantum systems, we propose quasiparticle-like formulae for $\widehat{\mathfrak{sl}(n)}_{k=1}$ characters.

The $\widehat{\mathfrak{sl}(2)}_{k=1}$ case is re-examined first. The novel formulation yields a direct connection to the fractional statistics in the short-range interacting model, and provides a clear description of the spinon character formula.

Generalizing the observation, we find formulae for $\widehat{\mathfrak{sl}(n)}_{k=1}$, which can be proved by the Durfee rectangle formula.

1. Introduction

Quasiparticles character formulae have been an issue of current interests [1–3, 7, 4–6]. They are proposed in various contexts; in the studies of lattice models, coset field theories or parafermionic field theories. These formulae have backgrounds in physics; they have originated from the description of particle contents in terms of string solutions to the Bethe ansatz equations (BAE).

There may be, however, many ways to interpret and, thereby represent, affine characters in term of quasiparticles. We refer to studies on long-range interacting models which provide a different point of view [8–13].

Recently, a novel approach to describe short-range interacting one-dimensional (1D) quantum systems at finite temperature has been developed [16–30]. In a sense, it offers different 'basis' from strings. One might naturally expect a novel way to represent affine characters in terms of these new basis. In this report, we will explicitly show that this expectation is indeed realized for affine level 1 $\mathfrak{sl}(n)$ (hereafter referred to as $\widehat{\mathfrak{sl}(n)}_{k=1}$). There could be several (quasi)particles in a physical system. Some of them describe the 'Dirac sea', and the others represent excitations. The standard string approach does not distinguish them, *a priori*. Furthermore, the arbitrary lengths of strings are allowed for the cases under consideration‡. Consequently, there exist infinite numbers of '(quasi)particles', which makes the description of the affine characters difficult in 'string basis' (except for the case where the numbers of (quasi)particles reduce finitely). In the novel approach, the Dirac sea is decoupled from others and the number of 'particles' is finite. Thus the affine character

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‡ Our main interest concerns vertex models, not RSOS models in this report.

can be represented by finite numbers of summation indices, which may be associated to quantum numbers of 'particles'.

This paper is organized as follows. We will first revisit the thermodynamics of the spin $\frac{1}{2}XXX$ model, i.e. the $\widehat{\mathfrak{sl}(2)}_{k=1}$ case. Some background in our approach will be reviewed, and reinterpreted. This simple example already provides the essence of our argument. The novel formulation yields a direct connection to the fractional statistics in this short-range interacting model. In a sense, we are directly dealing with excitations which reduce to spinons in the $T \to 0$ limit. We will see that the appropriate analytic continuation of the Rogers dilogarithm function leads to the 'spinon' character for this case. Generalizing the observation there, we will propose a quasiparticle-like character formula for $\widehat{\mathfrak{sl}(n)}_{k=1}$ in section 3. A brief summary and discussions are given in section 4.

2. $\widehat{\mathfrak{sl}(2)}_{k=1}$ revisited

The thermodynamics of the spin $\frac{1}{2}XXX$ model has a long history of success. The standard approach string hypothesis formulates it by the infinite numbers of unknowns $\eta_m(m = 1, \ldots, \infty)$ [14, 15]. As noted in the introduction, this is a drawback in presenting the affine character or partition function.

A different approach is initiated in [16] and subsequently progressed, especially in [22]. It integrates two important ingredients, the equivalence theorem between D-dimensional quantum systems and $(D + 1)$-dimensional classical systems, and the (Yang–Baxter) integrable structure. We start from a 1D quantum system at $T = 1/\beta$ with system size L. By introducing a fictitious dimension M, it can be mapped to a two-dimensional (2D) classical vertex model defined on an $L \times M$ square lattice. We first observe that the anisotropy in the coupling constant $u = -\beta/M$ 'intertwines' the finite-temperature system to a finite-size system. The partition function can be represented in terms of the 'quantum transfer matrix' (QTM) propagating in the crossing channel. The second observation is that the QTM constitutes a commuting family labelled by a complex parameter v. These observations can be summarized by the following formula for the partition function by the QTM,

$$\mathcal{Z}(\beta) = \lim_{L\to\infty} \lim_{M\to\infty} \mathrm{Tr}\left(T_{\mathrm{QTM}}\left(u = -\frac{\beta}{M}, v = 0\right)\right)^L .$$

The existence of parameter v actually leads to a deeper result. The BAE for diagonalization of the QTM can be recognized as the analyticity condition of $T_{\mathrm{QTM}}(u, v)$ in the complex v plane. The investigation of the eigenvalue $T_{\mathrm{QTM}}(u, v)$ then turns out to be that of the analyticity of auxiliary functions in the complex v-plane.

The explicit eigenvalue $\Lambda(u, v)$ of $T_{\mathrm{QTM}}(u, v)$ is composed of two terms $\lambda_1(u, v)$ and $\lambda_2(u, v)$

$$\Lambda(u, v) = \lambda_1(u, v) + \lambda_2(u, v)$$
$$\lambda_1(u, v) = \phi_-(v - \mathrm{i})\phi_+(v)\frac{Q_1(v + \mathrm{i})}{Q_1(v)} \qquad \lambda_2(u, v) = \phi_+(v + \mathrm{i})\phi_-(v)\frac{Q_1(v - \mathrm{i})}{Q_1(v)}$$

where

$$\phi_\pm(v) = (v \pm \mathrm{i}u)^{M/2} \qquad Q_1(v) = \prod_{j=1}^{m}(v - v_j) \tag{1}$$

and $\{v_j\}$ is the solution to the BAE,

$$-1 = \frac{\phi_-(v_j - \mathrm{i})\phi_+(v_j)}{\phi_+(v_j + \mathrm{i})\phi_-(v_j)} \frac{Q_1(v_j + \mathrm{i})}{Q_1(v_j - \mathrm{i})}.$$

We remark that each λ_i carries spurious singularities, $1/Q_1(v)$. The BAE condition, however, assures pole-freeness of the sum of them. Respecting this, the auxiliary functions are introduced in [22],

$$\begin{aligned} \mathfrak{a}^+(v) &= \frac{\lambda_1(u, v + \mathrm{i}\gamma)}{\lambda_2(u, v + \mathrm{i}\gamma)} \qquad & \mathfrak{A}^+(v) &= 1 + \mathfrak{a}^+(v) \\ \mathfrak{a}^-(v) &= \frac{\lambda_2(u, v - \mathrm{i}\gamma)}{\lambda_1(u, v - \mathrm{i}\gamma)} \qquad & \mathfrak{A}^-(v) &= 1 + \mathfrak{a}^-(v). \end{aligned} \tag{2}$$

The shift in arguments by $0 < \gamma \leqslant \frac{1}{2}$ is introduced to avoid zeros of $\mathfrak{A}^\pm$ on the real axis.

For the evaluation of the free energy, we only have to evaluate the largest eigenvalue of $T_{\mathrm{QTM}}(u, v)$ due to the existence of finite gap at finite temperatures. Let us summarize the relevant results in [22, 27]. The sector lies in $m = \frac{M}{2}$. The auxiliary functions have the following strips where they are analytic and nonzero and have constant asymptotic values (ANZC);

$$\begin{aligned} &\mathfrak{a}^+(v),\ -u - \gamma \leqslant \Im v \leqslant 1 + u - \gamma \qquad && \mathfrak{A}^+(v),\ -\gamma < \Im v < 1 - \gamma \\ &\mathfrak{a}^-(v),\ -1 - u + \gamma \leqslant \Im v \leqslant u + \gamma \qquad && \mathfrak{A}^-(v),\ -1 + \gamma < \Im v < \gamma. \end{aligned}$$

Note that u is a negative small quantity.

In particular, the ANZC properties of $\frac{\mathfrak{a}^\pm(v - \mathrm{i}\gamma)\phi_\mp(v)}{\phi_\pm(v)}$, $\frac{\mathfrak{A}^\pm(v - \mathrm{i}\gamma)\phi_\mp(v)}{Q_1(v)}$ near the real axis leads to two nonlinear integral equations,

$$\begin{aligned} \log \mathfrak{a}^+(v) &= -2\pi\beta\Psi(v + \mathrm{i}\gamma) + K * \log \mathfrak{A}^+(v) - K * \log \mathfrak{A}^-(v + 2\mathrm{i}\gamma) \\ \log \mathfrak{a}^-(v) &= 2\pi\beta\Psi(v - \mathrm{i}\gamma) + K * \log \mathfrak{A}^-(v) - K * \log \mathfrak{A}^+(v - 2\mathrm{i}\gamma) \end{aligned}$$

where

$$\Psi(v) = \frac{\mathrm{i}}{2\sinh \pi v} \qquad K(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kv}}{1 + \mathrm{e}^{|k|}}\, \mathrm{d}k \tag{3}$$

and $A * B(x)$ denotes convolution $\int_{-\infty}^{\infty} A(x - y)B(y)\mathrm{d}y$.

The free energy can be represented in an integral form using $\log \mathfrak{A}^\pm$.

$$-\beta f = 2\beta \log 2 + \int_{-\infty}^{\infty} \Psi(v + \mathrm{i}\gamma) * \log \mathfrak{A}^+(v)\, \mathrm{d}v - \int_{-\infty}^{\infty} \Psi(v - \mathrm{i}\gamma) * \log \mathfrak{A}^-(v)\, \mathrm{d}v. \tag{4}$$

The first term in the l.h.s. is the well known ground state energy. We thus manifestly separate the contribution by the Dirac sea. Thermal fluctuations are described by $\mathfrak{a}^\pm$. Although the integrands in (4) have fictitious dependence on γ, the result of integration is independent of its value. This comes from the ANZC properties of $\mathfrak{A}^+$ and $\mathfrak{A}^-$ within the strips described above. Note $\gamma \leqslant \frac{1}{2}$.

The relevant result for the present discussion is the following expression for the central charge.

$$c_0 = L\left(\frac{\mathfrak{a}^+(\infty)}{\mathfrak{A}^+(\infty)}\right) + L\left(\frac{\mathfrak{a}^-(\infty)}{\mathfrak{A}^-(\infty)}\right) \qquad (= 1) \tag{5}$$

where $L(z)$ is Rogers dilogarithm function

$$L(z) = -\frac{1}{2} \int_0^z \left(\frac{\log(1 - x)}{x} + \frac{\log(x)}{1 - x}\right) \mathrm{d}x \tag{6}$$

which has a branch cut along the lines $(-\infty, 0]$, $[1, \infty)$. The partition function can be, after subtraction of the ground state energy, approximated to

$$\mathcal{Z}(\beta) \sim q^{-c_0/24}$$

where $q = \exp(-4\pi/v_F\beta)$ and v_F is the Fermi velocity.

Now we reinterpret these results. First, we note a result from numerical investigation. Fix $\gamma = \frac{1}{2}$. For sufficiently low temperatures, one observes a crossover behaviour of $\mathfrak{a}$ in very narrow regions ($v \sim \pm \log \beta$). This justifies the approximation of the kernel function K in (3) to $\frac{1}{2}\delta(v)$. Assuming the validity of this approximation, we perform the change of variable $k + \pi/2 = 2\tan^{-1}(e^v)$. By writing $w_1(k) = 1/\mathfrak{a}^+$, $w_2(k) = 1/\mathfrak{a}^-$, the resultant equations can now be cast into the form,

$$\begin{aligned} &\exp(\beta\epsilon_a^0(k)) = (1 + w_a) \prod_{b=1,2} \left(\frac{w_b}{1 + w_b}\right)^{\mathfrak{g}_{b,a}} \\ &f = -T \sum_{a=1,2} \int G_a \log\left(1 + \frac{1}{w_a}\right) \mathrm{d}k \\ &\epsilon_a^0(k) = \cos k \qquad G_1 = G_2 = \tfrac{1}{2}. \end{aligned} \tag{7}$$

This is nothing but the equation describing particles obeying fractional exclusion statistics [31, 32], where the statistical interaction matrix is given by

$$\mathfrak{g} = \begin{pmatrix} \frac{1}{2}, & \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} \end{pmatrix}. \tag{8}$$

This coincides with known $\mathfrak{g}$ for spinons [33]. This is an explicit demonstration of the fractional statistics in the solvable spin chain model, although in an approximate sense. One recognizes a clear difference between the above result and the fractional exclusion statistics approach to the same model based on the conventional string picture. See the appendix of [34].

We further exploit this coincidence. Consider the conformal limit (or character limit). The higher terms are now contributing to $\mathcal{Z}$. The analysis of contributions from excited states is of considerable interest. See [35–41] for progress in related areas and [22, 29, 42] in the context of QTM. For T finite, it requires the explicit evaluation of zeros of auxiliary functions entering into the 'physical strip' which is free from zeros and singularities for the largest eigenvalue sector. This involves extensive numerical studies. See [22, 39, 40, 42] for example. Since our concern here is the asymptotic behaviours, we can avoid such numerics by following the strategy in [7]. By adopting a general contour $\mathcal{C}$ for the integration in (6), one can define an analytically continued dilogarithm function $L_{\mathcal{C}}(z)$. Such a contour-dependent dilogarithm function successfully recovered conformal spectra for lower excitations. We assume this in general; all excitation spectra in the conformal limit shall be described by an analytically continued dilogarithm function.

Let $\mathcal{C}$ be a contour starting from the origin and terminating at f, such that it crosses first $[1, \infty)$ η_1 times, then crosses $(-\infty, -1]$ ξ_1 times, then again η_2 times and so on. We adopt the convention $\eta_j, \xi_j < 0$ if the contour crosses branch cuts clockwise and $\eta_j, \xi_j > 0$ otherwise. Let us denote this by $\mathcal{C}[f|\xi_1, \xi_2, \ldots|\eta_1, \eta_2, \ldots]$. Then contours are parametrized by a set $\{\mathcal{S}\} = \{\mathcal{C}^{\pm}[f^{\pm}|\xi_m^{\pm}|\eta_m^{\pm}]\}$, where $f^{\pm} = \mathfrak{a}^{\pm}(\infty)/\mathfrak{A}^{\pm}(\infty)$. The effective central charge can be expressed by

$$\begin{aligned} &c(\mathcal{S}) = c_0 - 24\mathcal{T}(\mathcal{S}) \qquad \mathcal{T}(\mathcal{S}) = \tfrac{1}{2}{}^t\boldsymbol{n}\mathfrak{g}\boldsymbol{n} - \sum_{a=\pm}\sum_{j=1,\cdots} \xi_j^a(\eta_1^a + \cdots + \eta_j^a) \\ &\boldsymbol{n} = \begin{pmatrix} n^+ \\ n^- \end{pmatrix} \end{aligned} \tag{9}$$

where $n^{\pm} = \sum_j \eta_j^{\pm}$.

Following [7], we choose a subset $\mathcal{O}$ of all possible contours,

$$\{\mathcal{C}^{\pm}[f^{\pm}|\xi_1^{\pm}, \ldots, \xi_{n^{\pm}}^{\pm}|\overbrace{1, 1, \ldots, 1}^{n^{\pm}}, 0, \ldots,]\}$$

and $\xi_j^{\pm} \leqslant 0$, $j = 1, \ldots, n^{\pm}$.

The summation over $\mathcal{O}$ leads to the following 'partition function'.

$$\sum_{\mathcal{S}\in\mathcal{O}} q^{-c(\mathcal{S})/24} = q^{-c_0/24} \sum_{n^+,n^-} \frac{q^{1/2^t \boldsymbol{n} \mathfrak{g} \boldsymbol{n}}}{(q)_{n^+}(q)_{n^-}} \tag{10}$$

where $(q)_m = \prod_{i=1}^m (1-q^i)$. This is nothing but the spinon character formula. The explicit forms of affine characters have been obtained in mathematical literatures, e.g. [43]. Let α be the simple for $\mathfrak{sl}_2$ and $Q = \mathbb{Z}\alpha$. Then $\widehat{\mathfrak{sl}(2)}_{k=1}$ character reads,

$$\mathrm{ch}\ L(\Lambda)(q, u) = \frac{1}{\eta(q)} \sum_{\alpha\in Q} q^{(\alpha+\bar{\Lambda})^2/2} \mathrm{e}^{2\pi \mathrm{i}\langle \alpha+\bar{\Lambda}, u\rangle} \tag{11}$$

where $\bar{\Lambda}$ denotes the classical part and $\eta(q) = q^{1/24} \prod_{j\geqslant 1}(1 - q^j)$. We use the notation $z = \mathrm{e}^{2\pi\mathrm{i}\langle\alpha,u\rangle}$.

Our partition function may trace all energy levels and their degeneracies. Thus it should coincide with (11) after appropriate modification due to z. Actually the following theorem holds [3, 10, 44].

Theorem 1. The character for $\widehat{\mathfrak{sl}(2)}_{k=1}$ assumes the following fermionic form,

$$\mathrm{ch}\ L(\Lambda)(q, z) = q^{-c_0/24} \sum_{n^+,n^-} \frac{q^{1/2 \boldsymbol{n} \mathfrak{g} \boldsymbol{n}}}{(q)_{n^+}(q)_{n^-}} z^{1/2(n^+-n^-)}$$

where $n^+ + n^- =$ even (odd) for $\Lambda = \Lambda_0(\Lambda_1)$.

The fermionic character further implies that our auxiliary functions indeed describe the spinons [45, 46].

Let us summarize the lesson from $\widehat{\mathfrak{sl}(2)}_{k=1}$. We define auxiliary functions respecting the cancellation of spurious poles. These auxiliary functions satisfy coupled integral equations. In the 'character limit', the partition function is described by $\mathfrak{g}$ which can be readily evaluated form asymptotic behaviours of the coupled integral equations. Taking account of winding numbers of the dilogarithm function, we reach the quasiparticle character formula.

3. Novel character formula for $\widehat{\mathfrak{sl}(n)}_{k=1}$

Next we consider $\widehat{\mathfrak{sl}(n)}_{k=1}$ $n = r + 1$ arbitrary.

The eigenvalue of QTM consists of $r + 1$ terms,

$$\lambda_a(v) = \phi_a(v) \frac{Q^{(a-1)}(v - \mathrm{i}) Q^{(a)}(v + \mathrm{i})}{Q^{(a-1)}(v) Q^{(a)}(v)} \qquad (a = 1, \ldots, r+1)$$

$$\phi_a(v) = \begin{cases} a = 1 & \phi_+(v)\phi_-(v - \mathrm{i}) \\ a = r + 1 & \phi_+(v + \mathrm{i})\phi_-(v) \\ \text{otherwise} & \phi_+(v)\phi_-(v) \end{cases} \tag{12}$$

$$Q^{(a)}(v) = \begin{cases} a = 0 \text{ or } r + 1 & 1 \\ \text{otherwise} & \prod_j (v - v_j^{(a)}). \end{cases}$$

Neighbouring terms share common denominators which seem to bring about singularity. The BAE elucidates these fictitious poles,

$$\mathrm{Res}_{v=v_j^{(a)}}(\lambda_a(v)+\lambda_{a+1}(v))=0.$$

(For cancellations in the context of the analytic Bethe ansatz, see [47].)

More generally, in the string of terms $\lambda_{r+1}+\cdots+\lambda_i$, most of their 'singularities' are cancelled out except these from λ_i. These extra poles can be cancelled by adding λ_{i-1} which brings about other singularities. The proper choice of the auxiliary functions may respect this 'pole cancelling' property,

$$\begin{aligned}
\mathfrak{a}_i^+(v) &= \frac{\lambda_{r-i+1}(v+\mathrm{i}\gamma)}{(\lambda_{r+1}(v+\mathrm{i}\gamma)+\lambda_r(v+\mathrm{i}\gamma)+\cdots\lambda_{r-i+2}(v+\mathrm{i}\gamma))} \\
&\qquad \mathfrak{A}_i^+(v)=1+\mathfrak{a}_i^+(v), \quad (i=1,\ldots,r) \\
\mathfrak{a}_i^-(v) &= \frac{(\lambda_{r+1}(v-\mathrm{i}\gamma)+\lambda_r(v-\mathrm{i}\gamma)+\cdots\lambda_{r-i+2}(v-\mathrm{i}\gamma))}{\lambda_{r-i+1}(v-\mathrm{i}\gamma)} \\
&\qquad \mathfrak{A}_i^-(v)=1+\mathfrak{a}_i^-(v), \quad (i=1,\ldots,r).
\end{aligned} \tag{13}$$

These choices actually generalize the known results for $r=1$ in the previous section and $r=2$ [48]†. The straightforward generalization of the ANZC-type arguments leads to the following equation for the asymptotic values

$$\widetilde{\log}\mathfrak{a}(\infty)=\mathcal{K}\widetilde{\log}\mathfrak{A}(\infty)$$

where $\widetilde{\log}\mathfrak{a}$ is the abbreviation of the column vector, $(\log\mathfrak{a}_1^+,\ldots,\log\mathfrak{a}_r^+,\log\mathfrak{a}_1^-,\ldots,\log\mathfrak{a}_r^-)$ and similarly for $\widetilde{\log}\mathfrak{A}$. The generalization of $\mathfrak{g}$ in (8) is now explicitly given by

$$\mathfrak{g}=I-\mathcal{K}=\begin{pmatrix} C_{\mathfrak{sl}_{r+1}}^{-1}, & C_{\mathfrak{sl}_{r+1}}^{-1}(I-U) \\ (I-D)C_{\mathfrak{sl}_{r+1}}^{-1}, & (D-I)C_{\mathfrak{sl}_{r+1}}^{-1}(U-I)\end{pmatrix}$$

where $C_{\mathfrak{sl}_{r+1}}$, I denotes the Cartan matrix for $\mathfrak{sl}_{r+1}$ and the $r\times r$ identity matrix, respectively. D and U is defined by $D_{i,j}=\delta_{i,j+1}$, $U={}^tD$. Repeating similar arguments as in the case of $\widehat{\mathfrak{sl}(2)}_{k=1}$, we have the partition function,

$$\mathcal{Z}=q^{-r/24}\sum_{n_j^\pm\geqslant 0}\frac{q^{{}^t\boldsymbol{n}\mathfrak{g}\boldsymbol{n}/2}}{(q)_{\boldsymbol{n}}}$$

where

$${}^t\boldsymbol{n}=(n_1^+,\ldots,n_r^+,n_1^-,\ldots,n_r^-) \qquad (q)_{\boldsymbol{n}}=\prod_{j=1}^r (q)_{n_j^+}(q)_{n_j^-}. \tag{14}$$

Affine character for $\widehat{\mathfrak{sl}(r+1)}_{k=1}$ which generalizes equation (11) is given by

$$\mathrm{ch}\,L(\Lambda)(q,z_1,\ldots,z_r)=\frac{1}{\eta(q)^r}\sum_{\alpha\in Q}q^{(\alpha+\bar\Lambda)^2/2}\mathrm{e}^{2\pi\mathrm{i}\langle\alpha+\bar\Lambda,u\rangle} \tag{15}$$

where $Q=\sum_{i=1}^r\mathbb{Z}\alpha_i$, $z_i=\mathrm{e}^{2\pi\mathrm{i}\langle\alpha_i,u\rangle}$, $z^{\boldsymbol{n}}=\prod_{j=1}^r z_j^{n_j}$.

Let us present the main result in this report.

† I thank Andreas Klümper for the discussion on this point.

Theorem 2. Let $L(\Lambda_a)$ be the highest weight module for affine Lie algebra $\widehat{\mathfrak{sl}(r+1)}_{k=1}$ with the highest weight Λ_a. Then the character has a quasiparticle-like form,

$$\mathrm{ch}L(\Lambda_a)(q,z) = q^{-r/24} \sum_{n_j^\pm \geqslant 0} \frac{q^{{}^t\boldsymbol{n}\mathfrak{g}\boldsymbol{n}/2}}{(q)_{\boldsymbol{n}}} z^{\boldsymbol{\mu}(\boldsymbol{n})} \tag{16}$$

where the exponent of z reads

$$\begin{aligned} &\boldsymbol{\mu}(\boldsymbol{n}) = {}^t(\mathfrak{W}^{-1})C^{-1}_{\mathfrak{sl}_{r+1}}((I-D)^{-1}\boldsymbol{n}^+ - \boldsymbol{n}^-) \qquad \mathfrak{W} = \mathbb{C}_{\mathfrak{sl}_{r+1}}\mathfrak{W}^\vee\mathbb{C}^{-1}_{\mathfrak{sl}_{r+1}} \\ &\mathfrak{W}^\vee = S*(I-U)^{-1} \qquad (S*A)_{i,j} = A_{i,\tilde{j}} \qquad \tilde{j} = j+1 (\mathrm{Mod}\ r). \end{aligned} \tag{17}$$

The summation is subject to the rule; $(\mathfrak{g}\boldsymbol{n})_1 = 1 - \frac{a}{r+1} (\mathrm{Mod}\ \mathbb{Z})$.

For the proof of the above formula, we prepare some lemmas.

Lemma 1 ('Durfee rectangle formula').

$$\frac{1}{(q)_\infty} = \sum_{a,b\geqslant 0, a-b=m} \frac{q^{ab}}{(q)_a(q)_b}.$$

Here the difference m is an arbitrary but fixed integer.

We prove the equivalence between equation (15) and (16) by fixing weight $\mu(\boldsymbol{n})$. This allows us to rewrite the equations in term of $\{n_j^-\}$, as it imposes relations between $\{n_j^+\}$ and $\{n_j^-\}$;

$$\boldsymbol{n}^+ = (I-D)(\boldsymbol{n}^- + \nu) \qquad \text{where } \nu = C_{\mathfrak{sl}_{r+1}}{}^t\mathfrak{W}\mu(\boldsymbol{n}). \tag{18}$$

Lemma 2. Under equation (18), the q-exponent in equation (16) can be expressed as

$$\begin{aligned} &{}^t\boldsymbol{n}\mathfrak{g}\boldsymbol{n}/2 = \mathcal{F}(\nu) + \mathcal{G}(\nu, n^-) \\ &\mathcal{F}(\nu) = {}^t\boldsymbol{\nu}(I-U)C^{-1}_{\mathfrak{sl}_{r+1}}(I-D)\boldsymbol{\nu}/2 \\ &\mathcal{G}(\nu) = \tfrac{1}{2}({}^t\boldsymbol{\nu}(I-U)\boldsymbol{n}^- + {}^t\boldsymbol{n}^-(I-D)\boldsymbol{\nu} + {}^t\boldsymbol{n}^- C_{\mathfrak{sl}_{r+1}}\boldsymbol{n}^-). \end{aligned} \tag{19}$$

This can be easily derived with the aid of the trivial relation,

$$C_{\mathfrak{sl}_{r+1}} = (I-U) + (I-D). \tag{20}$$

The summation condition is now simplified and independent of n^-. To be precise, the following lemma holds.

Lemma 3. The summation condition, $(\mathfrak{g}\boldsymbol{n})_1 = 1 - \frac{a}{r+1} (\mathrm{Mod}\ \mathbb{Z})$ is equivalent to $\nu_1 + \cdots + \nu_r = r+1-a (\mathrm{Mod}\ r+1)$. The latter condition is indeed satisfied by equation (18).

A straightforward calculation shows that the original summation condition reduces to $n_1 - 1 = (a + \nu_1 + \cdots + \nu_r)/(r+1) (\mathrm{Mod}\ \mathbb{Z})$, which proves the former part of the lemma. The latter part can also be easily shown by noticing that α in equation (15) is a member of Q. Let $\alpha = \sum m_i\alpha_i, m_i \in \mathbb{Z}$. Equating the exponent of z in equations (15) and (16), we have $\nu = \boldsymbol{m} + C^{-1}_{\mathfrak{sl}_{r+1}}\boldsymbol{e}_a$, where $\boldsymbol{e}_a$ denotes the unit vector with only its ath entry unity and all others zero. By substituting this into equation (18), we find $\nu_1 + \cdots + \nu_r = (r+1)m_1 + r + 1 - a$. Thus the latter part is proved.

Lemma 4. In terms of $\{m_i\}$ defined in the above, we have

$$\mathcal{F} = {}^t\boldsymbol{m}C_{\mathfrak{sl}_{r+1}}\boldsymbol{m} + 2m_a + (C^{-1}_{\mathfrak{sl}_{r+1}})_{a,a}. \tag{21}$$

This is a consequence of the relation

$$C^{-1}_{\mathfrak{sl}_{r+1}} = \mathfrak{W}^{\vee}(I-U)C^{-1}_{\mathfrak{sl}_{r+1}}(I-D)^t(\mathfrak{W}^{\vee})$$

which can be shown by a direct calculation.

Proof of theorem 2. We first fix μ. Due to lemmas 3 and 4, for each fixed μ, equation (16) is now rewritten as

$$q^{-r/24}z^{\mu}q^{\mathcal{F}}\sum_{n_j^-}\frac{q^{\mathcal{G}}}{(q)_{\boldsymbol{n}}}$$

where $\{n_j^+\}$ in the denominator are functions of $\{n_j^-\}$ by equation (18). Thanks to lemma 4, the summations over $\{n_j^-\}$ are almost non-restricted except for 'implicit' conditions, $n_j^+ \geqslant 0$. This actually meets the condition of lemma 1. Let us see this explicitly for the summation over n_1^-. Note first that $\mathcal{G} = (n_1^-)^2 - n_1^- n_2^- + n_1^-(\nu_1-\nu_2)+$ (terms independent of n_1^-), and $n_1^+ = n_1^- - n_2^- + \nu_1 - \nu_2$. Thus the contribution from the n_1^- summation reduces to

$$\sum_{n_1^- \geqslant n_1^*}\frac{q^{(n_1^-)^2 - n_1^- n_2^- + n_1^-(\nu_1-\nu_2)}}{(q)_{n_1^-}(q)_{n_1^- - n_2^- + \nu_1 - \nu_2}}$$

where $n_1^* := \max(0, n_2^- + \nu_2 - \nu_1)$. Identifying $a = n_1^-, b = n_1^- - n_2^- + \nu_1 - \nu_2, m = -n_2^- + \nu_1 - \nu_2$, we can apply lemma 1 straightforwardly. Thus the contribution from the n_1^- part is simply given by $1/(q)_\infty$.

Repeating this successively we reach the expression, $\frac{z^{\mu}q^{\mathcal{F}}}{\eta(q)^r}$. By lemma 4, this can be entirely written in term of $\{m\}$ as,

$$\frac{z^{\boldsymbol{m}+C^{-1}_{\mathfrak{sl}_{r+1}}e_a}q^{{}^t\boldsymbol{m}C_{\mathfrak{sl}_{r+1}}\boldsymbol{m}+2m_a+(C^{-1}_{\mathfrak{sl}_{r+1}})_{a,a}}}{\eta(q)^r}$$

which agrees with the term in equation (15) with $\alpha = \sum m_i\alpha_i$ and $\Lambda = \Lambda_a$. $\square$

As demonstrated in $\widehat{\mathfrak{sl}(2)}_{k=1}$, the Dirac sea contribution is separated and $\{\mathfrak{a}_i^{\pm}\}$ correspond to thermal excitations. Thus we may expect that they describe physical particles.

4. Summary and discussion

In this report, we present a character formula of the fermionic type for $\widehat{\mathfrak{sl}(r+1)}_{k=1}$. It has originated from the novel formulation of thermodynamics of 1D quantum systems at finite temperatures.

The interpretation of the resultant formula in terms of particles, however, poses problems. By coincidence for the $r=1$ case, we expect relations to $su(r+1)$ spinons [49]. Unfortunately, this seems to be false. For an example, the $su(3)$ spinon character needs six summation variables (or three) [49, 11] in contrast to the four variables in the present result.

We may assume there exist 'particles' $k^{(\pm)}$ corresponding to 'quantum numbers' $n_k^{\pm}$.

From (16), we identify their weights as in table 1.

'Particle' $a^{(+)}$ carries the weight in the highest weight module $V(\Lambda_a)$, while $a^{(-)}$ has the one in $V(\Lambda_1)$. In a sense, $\pm$'particles' are quite asymmetric, which does not seem to

Table 1.

Particle	Classical weight
$1^{(+)}$	Λ_1
$a^{(+)}$ $(2 \leqslant a \leqslant r-1)$	$\Lambda_{a+1} - \Lambda_1$
$r^{(+)}$	$\Lambda_1 - \Lambda_2$
$a^{(-)}$ $(1 \leqslant a \leqslant r-1)$	$\Lambda_{a+2} - \Lambda_{a+1}$
$r^{(-)}$	$\Lambda_2 - \Lambda_1$

be natural. The vertex operators associated to negative or positive roots bring a similar but 'symmetric' q-character for whole vacuum module [50]

$$\sum_{n_1^{\pm},\dots,n_r^{\pm} \geqslant 0} \frac{q^{\sum_{j,k}(n_j^+ - n_j^-)(C^{-1}_{\mathfrak{sl}_{r+1}})_{j,k}(n_k^+ - n_k^-) + \sum_j n_j^+ n_j^-}}{(q)_{\boldsymbol{n}}} \tag{22}$$

under r constraints, $\sum_k (C^{-1}_{\mathfrak{sl}_{r+1}})_{j,k}(n_k^+ - n_k^-) \in \mathbb{Z}$, $(j = 1, \dots, r)$. This formula also follows from a specialization of equation (15) using lemma 1. This similarity makes us expect that the vertex operator description of particles in equation (16) would provide an interesting future problem.

Obviously, several generalizations await us, such as the extension to higher levels or to other affine Lie algebras. The higher level cases correspond to fusion models in the terminology of lattice models. The problem which shares technical similarity to the present one, the finite-size correction study, has been successfully investigated *only* up to level 2 $\widehat{\mathfrak{sl}(2)}_{k=2}$. The result there suggests more elaborated combinations of λ_i are necessary for auxiliary functions [51, 52]. For general cases, the strategy for suitable choices of auxiliary functions are yet to be known. We hope that the present result triggers further investigations of these general cases, which should be interesting in view of the original problem of thermodynamics.

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